Automated Parameter Identification and Order Reduction for Discrete Time Series Models

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The development of an algorithm for the parameter identification of discrete time-series models for dynamic structural systems is presented. The estimated models are used to predict the transient response of structures subject to arbitrary input. The linear, structural transfer function is expressed in the form of an autoregressive moving-average (ARMA) discrete time-series model. The accurate estimation of the system parameters is often limited due to the influence of measurement noise. Overspecification of the system order has proven useful in reducing the bias errors, but the resulting model may be excessively large and will contain noise related modes. A completely automated approach for parameter identification and model reduction is presented. Experimental response data from a subscale sailplane is used to demonstrate the technique. The resulting reduced order models provided accurate prediction of the transient response of the sailplane to arbitrary inputs.

Nomenclature

a_i = autoregressive parameter
A_m = matrix of measured values (forward model estimation)
$A(z)$ = characteristic polynomial
b_i = moving-average parameter
$B(z)$ = moving-average polynomial
c_i = backward autoregressive parameter
$\boldsymbol{c}$ = vector of backward autoregressive parameters
$\boldsymbol{C}_m$ = matrix of measured values (backward model estimation)
$C(z)$ = backward characteristic polynomial
$C'(z)$ = conjugate backward characteristic polynomial
$C_1(z)$ = fixed factor of $C(z)$
$C_2(z)$ = varying factor of $C(z)$
d_i = backward moving-average parameter
$\boldsymbol{e}$ = residual vector
$e(k)$ = residual at time k
$f_i(z)$ = factor of $C(z)$ not containing the ith root of $C_2(z)$
r_i = residue of model at ith pole
$u(k)$ = discrete time value of the input
u_k = part of the singular value decomposition
$u_m(k)$ = measured discrete time value of the input
v_k = part of the singular value decomposition
$y(k)$ = discrete time value of the response
$y_m(k)$ = measured time value of the response
$\boldsymbol{y}_m$ = measured response vector (forward model estimation)
$\boldsymbol{y}_m'$ = measured response vector (backward model estimation)
z = forward shift operator
z_i = ith pole of the model
σ_k = kth singular value
$\boldsymbol{\theta}$ = parameter vector
$\boldsymbol{\theta}_{LS}$ = least-squares estimate of the parameter vector
$\boldsymbol{\theta}'$ = backward parameter vector
ζ = limit variable used to calculate residues

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Introduction

DYNAMIC system models of structures are developed for a variety of reasons. A model may be needed to implement a control algorithm or the model may be needed to determine if the system's response to a specific input will be acceptable. If the response exceeds allowable levels, either the structure is modified, or the input has to be altered. The modeling problem can be further complicated since, during service, certain types of structures can assume many different configurations. Mass loadings change, structural properties can vary with temperature, and localized component failure may occur. The many possible configurations during service of some structures place limits on the usefulness of certain analytic or numerical modeling procedures such as finite element analysis.

Corrected or updated models that are suitable for the rapid and accurate prediction of transient dynamic response for a specific set of loading conditions are often necessary. Recent developments in digital data acquisition and computation procedures provide the opportunity to "intelligently," rapidly, and automatically develop models from measured system response during normal system operation. Consider the problem of an airplane about to traverse a rough or damaged runway. A prediction of acceleration levels at several critical locations on the structure may determine whether the aircraft would be damaged while using the given runway or could help to determine the speed and path the aircraft should take to minimize the risk of damage. A model relating the vibration at the critical locations within the structure to the specific runway profile would be required. Significant changes in the dynamic characteristics of the aircraft can occur as its configuration changes; for example, the payload or fuel loading may be modified. When the structural configuration changes, the predictive model must change. This paper explores an approach to this problem in which digital time histories of the input and system output are acquired and the system model or transfer function automatically derived or updated.

For linear systems, discrete time-series models are well-suited for automatic digital identification. One such model that is attractive for this application because it is a minimum parameter model is the autoregressive moving-average model (ARMA). The basic ARMA model for a single input and single output (SISO) system has the form

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \cdots - a_p y(k-p) + b_0 u(k) + b_1 u(k-1) + \cdots + b_p u(k-p) \quad (1)$$

where $y(k)$ is the system response at the kth discrete time, $u(k)$ is the system input at the kth discrete time, p is the order or size of the model, and a and the b are the model's parameters. The response $y(k)$ can be a displacement, velocity, or acceleration of a specific point on the structure. The ARMA model has been used in numerous structural applications such as substructure modeling,[1] spectral estimation,[2] and vibration prediction for both discrete and continuous system.[3] The advantages of the model are its simple form, the limited number of system parameters, and the fact that the states of the model are directly measurable. Parameter-identification procedures have been developed which use digitally acquired time series data without conversion to another domain. Once system parameters have been identified, response prediction is a relatively easy process requiring only sequential multiplications and additions. The main limitations associated with the use of the ARMA model are the determination of the order of the system and the parameter identification in the presence of measurement noise. In practice, measurements of the applied input and output from the structure will be contaminated with measurement noise. The noise may be systematic or random but it is known to lead to biased parameter estimates.

This paper is concerned with the application of discrete time-series models for structural response prediction and will be limited to SISO applications. The paper presents a brief section that describes the ARMA model. The difficulties arising from the influence of measurement noise in the identification process and an algorithm that automatically addresses the bias problem are presented. Experimental results using a subscale sailplane illustrate the improvement of the algorithm over previous approaches.

Background

As indicated by Eq. (1), a state of the system, $y(k)$, can be expressed in terms of a sum of scaled previous values of the state and scaled values of input to the system. If there is no input to the structure, which is the case during free response, ARMA model reduces to its homogeneous form:

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \cdots - a_p y(k-p) \quad (2)$$

which is an autoregressive (AR) model. Rearranging the AR model in shift-operator notation and multiplying the equation by z^p yields

$$A(z)y(k) = [z^p + a_1 z^{p-1} + \cdots + a_p]y(k) = 0 \quad (3)$$

The roots of $A(z)$ in the z domain can be mapped into the Laplace domain and used to estimate natural frequencies and damping factors for the system. The nonhomogeneous part of the ARMA model relates the system response to the input time series and is referred to as the moving average (MA) part of the model. The accuracy of ARMA model for response prediction for both discrete and continuous structural systems has been shown in Ref. 3.

The goal of the identification scheme is to estimate the model parameters from measured time series, so that the resulting model can provide accurate system-response prediction to an arbitrary input. The ARMA model can be extended and written in vector form, using the measured time series, as

$$y_m(k) = [-y_m(k-1) - y_m(k-2)\cdots - y_m(k-p)u_m(k)u_m(k-1)\cdots u_m \times (k-p)]\boldsymbol{\Theta} + e(k) \quad (4)$$

where

$$\boldsymbol{\Theta} = [a_1 a_2 \cdots a_p b_0 b_1 \cdots b_p]^T \quad (5)$$

and where the subscript m denotes a measured or noise-corrupted quantity, $\boldsymbol{\Theta}$ is the parameter vector, and $e(k)$ is the residual error due to the measurement errors. For any continuous structure, the system order is infinite although there are typically a finite number of modes in the frequency range of interest. The band-limited ARMA model order is determined by the selection of the parameter p, where ideally p is twice the number of modes in the desired frequency range.

Given a set of measured input and output time-series data at N instances of time, $(N-p)$ equations can be formed by varying k in Eq. (4):

$$\boldsymbol{A}_m\boldsymbol{\Theta} = \boldsymbol{y}_m + \boldsymbol{e} \quad (6)$$

where

$$\mathbf{y}_m = [y_m(p+1)y_m(p+2)\cdots y_m(N)]^T \quad (7)$$

$$\boldsymbol{e} = [e(p+1)e(p+2)\cdots e(N)]^T \quad (8)$$

and

$$\boldsymbol{A}_m = \begin{bmatrix} -y_m(p) & \cdots & -y_m(1) & u_m(p+1)\cdots & u_m(1) \\ -y_m(p+1)\cdots & & -y_m(2) & u_m(p+2)\cdots & u_m(2) \\ \cdot & \cdots & \cdot & \cdot \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot \cdots & \cdot \\ -y_m(N-1)\cdots & -y_m(N-p) & u_m(N) & \cdots u_m(N-p) & \end{bmatrix} \quad (9)$$

For an ideal linear system, if the measurements were exact and $(N-p)$ equals $(2p+1)$, the residual errors would be zero and the parameters could be determined from the solution of Eq. (6). Because of the measurement errors, the most straightforward method of estimating the parameter vector $\boldsymbol{\Theta}$ is to choose $(N-p)$ greater than $(2p+1)$ and solve an overdetermined set equations in least-squares (LS) sense. The LS estimate is obtained by minimizing the square of the residuals. The estimated parameter vector is

$$\boldsymbol{\Theta}_{LS} = (\boldsymbol{A}_m{}^T\boldsymbol{A}_m)^{-1}\boldsymbol{A}_m{}^T\boldsymbol{y}_m \quad (10)$$

It has been well-documented that the LS parameter estimates are biased[4,5] due to the measurement errors.

There are many identification schemes that attempt to reduce estimation bias using iterative procedures. One of the most effective methods to reduce the bias is to simply overspecify the system order and use the LS algorithm. Overspecification has been used in time-domain modal analysis techniques to reduce the estimation bias.[6,7] Although overspecification of the ARMA model has proven useful in reducing noise bias, it also introduces problems. Consider the ARMA model expressed in shift-operator form after multiplying the numerator and denominator polynomial by z^p.

$$y(k) = \frac{b_0 z^p + b_1 z^{p-1} + \cdots + b_p}{z^p + a_1 z^{p-1} + \cdots + a_p} u(k) = \frac{B(z)}{A(z)} u(k) \quad (11)$$

The roots of the denominator and numerator of the transfer function represent the poles and zeros of the system. (Note the form of the model in Eq. (1) was chosen so that there are the same number of zeros as there are poles for the model.) For overspecified systems, the order of both the numerator and denominator exceeds that needed to describe the system. The additional poles and zeros introduced by overspecification can be referred to as computational poles and zeros. Overspecification can be viewed as multiplying both the numerator and denominator of the transfer function by computional polynomials. If the data are noise-free, the computational polynomi-

als will cancel. If the data contain noise, the computational polynomials will be perturbed so that they differ. The computational portions are included in the identification so that the noise bias will influence the location of the computational poles and zeros and the perturbation of the system poles will be reduced.

The overspecified model can be used to effectively predict vibration in many cases. But in some cases, the computational portion of the model can lead to inaccurate predictions. Another difficulty with the overspecified model is the fact that it requires additional terms in the prediction which slows computation. It would be advantageous to remove the computational portion from the model and reduce the model to its proper order. The overspecified model can be expressed through the use of partial-fraction expansion as

$$\left(\frac{B(z)}{A(z)}\right)_{\text{overspecified}} = \left(\frac{B(z)}{A(z)}\right)_{\text{system}} + \left(\frac{B(z)}{A(z)}\right)_{\text{computational}} \tag{12}$$

Time-domain modal techniques use modal confidence factors to separate the system and computational portions of the estimated modal model.[6,7] There has been no comparable automatic method to separate the system and computational portions of the overspecified LS ARMA model.

Reduced Backward Method

The overspecified LS ARMA model can be split into single-degree-of-freedom (modal) components using the partial-fraction expansion. If one could identify the system's natural frequencies, then the system portions of the model can be identified, the computational components eliminated, and the model order reduced. Knowledge of the system frequencies would be available from traditional modal methods or from time-domain methods utilizing free response or impulse response. But these sources of information are traditionally "off-line" methods. An automated method is proposed which would require only the input-output time histories. The method, the reduced backward method, was developed to provide this automation. The method uses overspecification to reduce noise bias, but the initial form of the ARMA model is written backward in time to help distinguish computational poles from system poles. Once the parameters of the backward model are known, the model can be rewritten in the forward direction. Since the system poles will be known, the residues of the overspecified transfer function can be used to formulate a reduced-order model.

The method estimates the parameters of an overspecified ARMA model written in the backward direction. The model is

$$y(k) = -c_1 y(k+1) - c_2 y(k+2) - \cdots - c_L\, y(k+L) + d_0 u(k) + d_1 u(k+1) + \cdots + d_L u(k+L) \tag{13}$$

where L is greater than p. The model is derived from an overspecified form of Eq. (1) by simple rearrangement. The model is referred to as the backward model because it appears to be predicting response in a backward direction, that is, the response is a function of future states of the response and the input. The model is acausal, and must be converted into a causal model before it would be useful as a predictive tool. The characteristic polynomial of the backward model is

$$C(z) = 1 + c_1 z + \cdots + c_L z^L \tag{14}$$

The advantage of the backward model is that the p stable roots of its characteristic equation are located outside the unit circle in the z domain. These p roots are determined by the structural system characteristics. There are $(L-p)$ extra computational roots whose locations may be anywhere. If the parameters c_i could be identified so that the computational poles of the backward model are forced inside the unit circle, then the system poles can be distinguished from them and the model can be reduced.

Given a set of input-output measurements $y_m(k)$ and $u_m(k)$, an overdetermined set of equations for the backward model can be written

$$\boldsymbol{C}_m \boldsymbol{\Theta}' = \boldsymbol{y}_m' + \boldsymbol{e} \tag{15}$$

where

$$\boldsymbol{\Theta}' = [c_1 c_2 \cdots c_L d_0 d_1 \cdots d_L]^T \tag{16}$$

$$\boldsymbol{y}_m' = [y_m(1) y_m(2) \cdots y_m(N-L)]^T \tag{17}$$

$$\boldsymbol{e}' = [e(1) e(2) \cdots e(N-L)]^T \tag{18}$$

and

$$\boldsymbol{C}_m = \begin{bmatrix} -y_m(2) & \cdots & -y_m(L+1) & u_m(1) & \cdots & u_m(L+1) \\ -y_m(3) & \cdots & -y_m(L+2) & u_m(2) & \cdots & u_m(L+2) \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ -y_m(N-L+1) & \cdots & -y_m(N) & u_m(N-L) & \cdots & u_m(N) \end{bmatrix} \tag{19}$$

For an ideal linear system with no noise in the data, Eq. (15) represents a singular problem since the problem has been overspecified. There are an infinite number of solutions for the noiseless case. The infinite number of solutions are the result of the infinite number of possible computational parts of the model. The minimum norm solution is the one solution whose vector length is a minimum. As is shown in the Appendix, the minimum norm solution forces the computational poles inside the unit circle.

In all practical cases, there is noise in the data and although the solution of Eq. (15) is possibly ill-conditioned, it should not be singular. One would like to develop a solution to Eq. (15) that had properties similar to the minimum norm solution of the noise-free problem. If the model is grossly overspecified, the solution of Eq. (15) will be highly ill-conditioned and a truncated singular value solution of the normal equations will approximate the minimum norm solution.[8] The normal equations are

$$(\boldsymbol{C}_m{}^T \boldsymbol{C}_m)\boldsymbol{\Theta}' = \boldsymbol{C}_m{}^T \boldsymbol{y}_m' \tag{20}$$

The coefficient matrix of the parameter vector $\boldsymbol{\Theta}'$ will be termed the backward correlation matrix. The truncated solution is

$$\boldsymbol{\Theta}' = \sum_{k=1}^{q} \sigma_k^{-1} [\boldsymbol{u}_k{}^T \boldsymbol{C}_m{}^T \boldsymbol{y}_m'] \boldsymbol{v}_k \tag{21}$$

where σ_k, $\boldsymbol{u}_k$, and $\boldsymbol{v}_k$, are q principal singular values and eigenvectors of the singular value decomposition of the backward correlation matrix $(\boldsymbol{C}_m{}^T \boldsymbol{C}_m)$. There should be $(2p+1)$ principal singular values for the ideal noise-free case, but when there is noise, it is often difficult to determine the number of the principal singular values. Some ways to estimate the number of principal singular values are discussed by Refs. 9 and 10. Often all of the singular values must be included and the solution then corresponds to the LS estimate of the backward coefficients. The method used in the remain-

der of the paper uses all of the singular values. It should be noted that if at all possible, the truncated solution should be used since it will come closer to approximating the minimum norm solution.

Once the LS estimates of the backward models are known, the model can be rewritten in the forward direction and expanded using partial fractions. Those portions corresponding to the computational modes can be eliminated. Then the reduced model can be assembled according to

$$\left(\frac{B(z)}{A(z)}\right)_{\text{system}} = \sum_{i=1}^{p} \frac{r_i}{z - z_i} \tag{22}$$

where the summation over i represents a sum of the system portions of the partial-fraction expansion, and r_i represents the residues of the overspecified backward model at the system pole locations:

$$r_i = \lim_{\zeta \to z_i} \left((z - \zeta) \frac{B(z)}{A(z)} \right) \tag{23}$$

(Note that when the model is written in the forward direction, the system poles are located outside the unit circle and the computational poles are also located outside the unit circle.)

Summary of Method

The computational steps in the reduced backward method can be summarized as follows:

1) Collect the time-series data.

2) Select the size of the overspecified backward model by choosing L. An arbitrarily large value can be used, since the method will automatically reduce the model to its proper order.

3) Form the backward correlation matrix $\boldsymbol{C}_m{}^T\boldsymbol{C}_m$ and the vector $\boldsymbol{C}_m{}^T\boldsymbol{y}_m'$ directly from the time-series data. The large matrix $\boldsymbol{C}_m(N - L \times 2L + 1)$ does not need to be formed.

4) Decompose the backward correlation matrix using singular value decomposition.

5) Estimate the value of q using the singular values of the backward correlation matrix. If an estimate cannot be made, set q equal to $(2L + 1)$. (This is the approach taken in this paper.)

6) Estimate the parameter vector $\boldsymbol{\theta}'$ according to Eq. (21).

7) Rewrite the estimated model in the forward direction.

8) Calculate the poles of the estimated forward model.

9) Sort the poles as either system poles (magnitude less than one) or computational poles (magnitude greater than one).

10) Calculate the residues of the estimated forward model at the system poles according to Eq. (23).

11) Reconstruct the reduced model using the system poles and residues according to Eq. (22).

Experimental Verification and Results

A subscale sailplane was used to demonstrate the reduced backward method. The sailplane was fabricated from fiberglass, wood, and plastic. It weighed 3.55 lb and had a wingspan of 128 in. and fuselage length of 50 in. The plane was rigidly mounted to a shaker at its center of gravity. The shaker was driven to simulate a rough runway. The system input was measured by an accelerometer mounted on the plane directly over the shaker and the system output was measured by another accelerometer on the wingtip. The shaker was driven by a pseudorandom binary input generated by computer. The input and output time series were filtered by two identical four-pole Butterworth antialiasing analog filters with low-pass cutoff frequencies set at 50 Hz. The digital time series were acquired using a Scientific Atlantic SD380 dynamic signal analyzer, at a sample rate of 256 Hz, and then transferred to a microcomputer for processing. Figure 1 shows the schematic of the experimental setup.

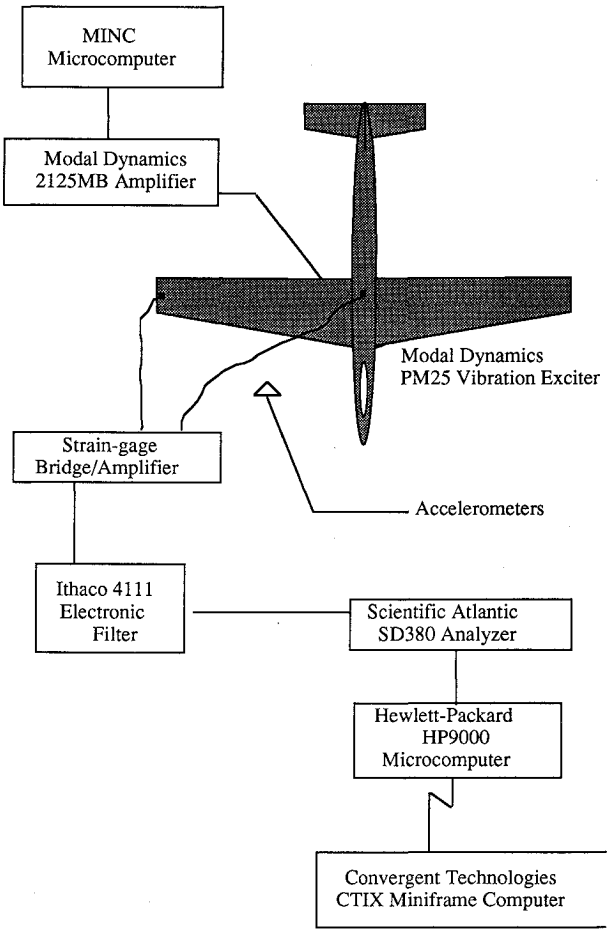

Fig. 1 Schematic of experiment.

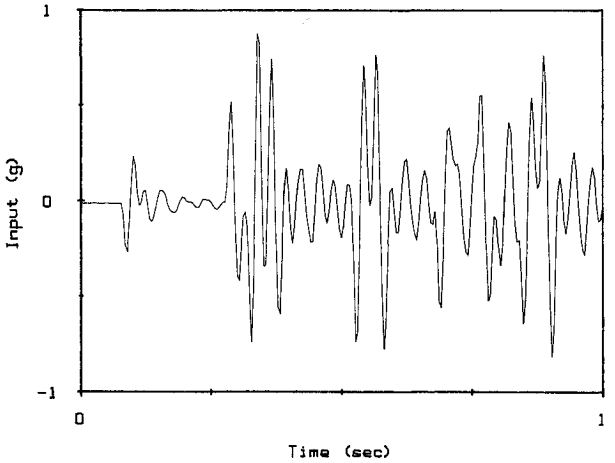

Fig. 2 Input to the sailplane (case A).

A portion of the measured input and output accelerations (case A) are shown in Figs. 2 and 3 (512 data pts, 2 s total time record). A second set of input and output accelerations (case B) were acquired and stored (512 data pts, 2 s total time record). A portion of the input acceleration for case B is shown in Fig. 4. The algorithm discussed above used the input and output from case A to estimate the ARMA model for this system. The ARMA model was then used with the input from case B to predict the output from case B without knowledge of the actual output. The predicted output could then be compared to the actual measured output in order to evaluate the identification procedure.

Figure 5 shows the singular value spectrum of the backward correlation matrix for a model overspecified to an order of 40.

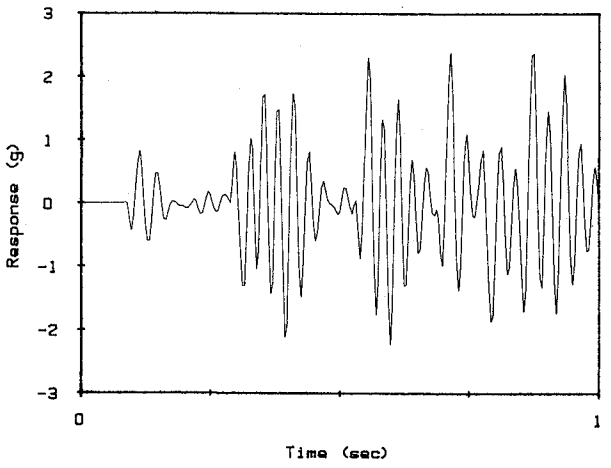

Fig. 3 Acceleration response of the sailplane (case A).

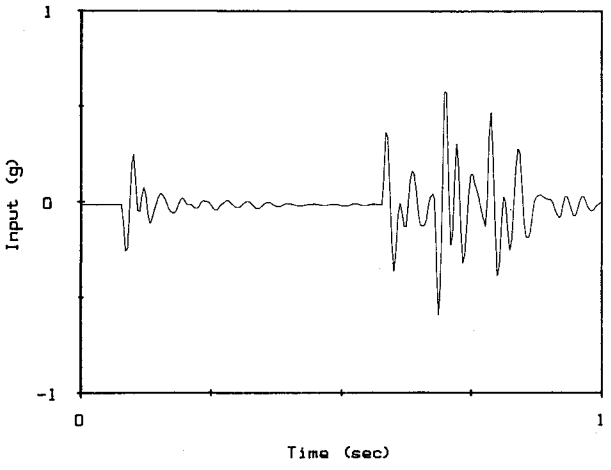

Fig. 4 Input to the sailplane (case B).

There are 81 parameters in the overspecified model and thus 81 singular values for the backward correlation matrix. Inspection of the singular values does not provide additional insight into the system order. The reduced backward method used all of the singular values in the computation of the overspecified backward model. The locations of the poles for the backward model in the z domain are shown in Fig. 6. There are seven pairs of complex poles located outside the unit circle corresponding to system poles. The system poles are close to the unit circle because they are lightly damped. These seven pairs were automatically selected to assemble the reduced-order model. The result is a 14th order model. The total computational time required to compute the reduced model on a MacIntosh II microcomputer (Motorola 68020 processor/ Motorola 68881 coprocessor) was 242 s. Figure 7 shows the magnitude frequency response function of the reduced-order model.

A frequency-domain estimate of the magnitude frequency response function was computed using the fast Fourier transform capabilities of the SD380 dynamic signal analyzer. The response function estimated using the same 2 s of data is shown in Fig. 8. The estimate is nonparametric and is the ratio of the discrete Fourier transforms (DFT) of the output and the input. Using exactly the same data, the reduced backward method was able to estimate the parametric form of the structural transfer function with a significant reduction in the effect of the noise. Frequency-domain methods usually require an ensemble average of transfer functions to help eliminate the influence of measurement noise. Figure 9 shows an average of 25 magnitude frequency response functions. Much of the noise has been eliminated through the averaging process and Fig. 9 closely resembles Fig. 7. There are major differences in the way the two figures were developed. Figure 7 was produced from a parametric model identified in the time domain from 2 s of measured data using the method described in this paper. Figure 9 was produced from 25 sets of 2 s of data and is an interim step in any frequency-domain method. A parametric frequency-domain model would still have to be developed, often requiring much user interaction, in order to arrive at a model suitable for response prediction.

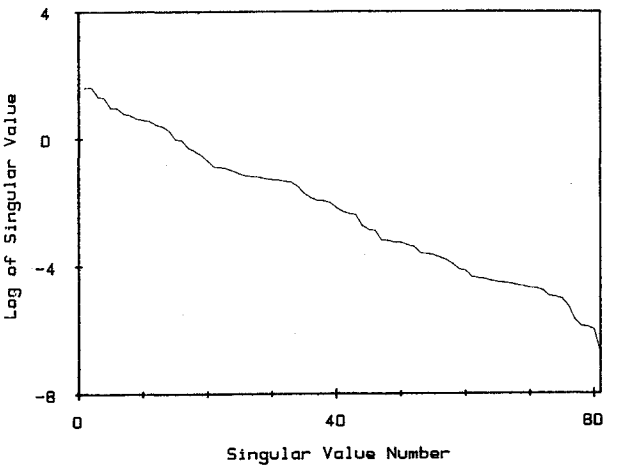

Fig. 5 Singular value spectrum of the backward correlation matrix.

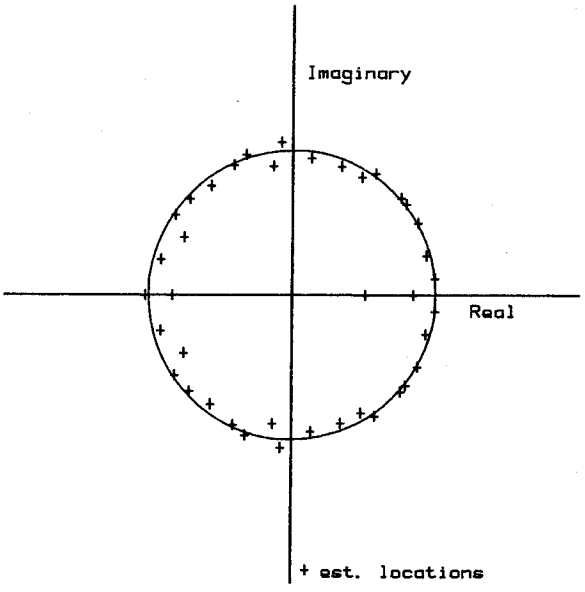

Fig. 6 Pole locations of the estimated backward model.

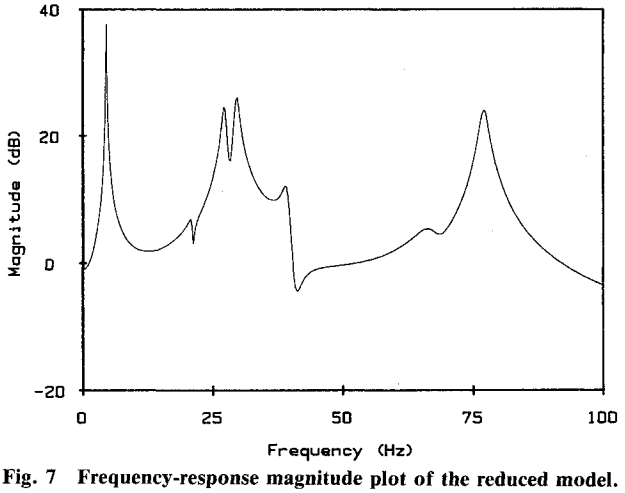

Fig. 7 Frequency-response magnitude plot of the reduced model.

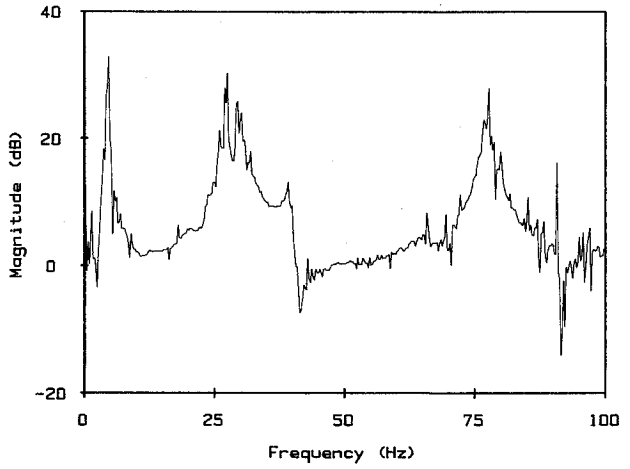

Fig. 8 Frequency-response function from frequency-domain techniques.

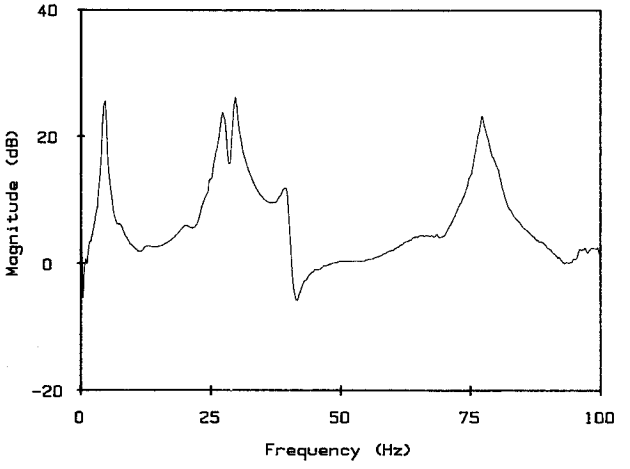

Fig. 9 Averaged frequency-response function from frequency-domain techniques.

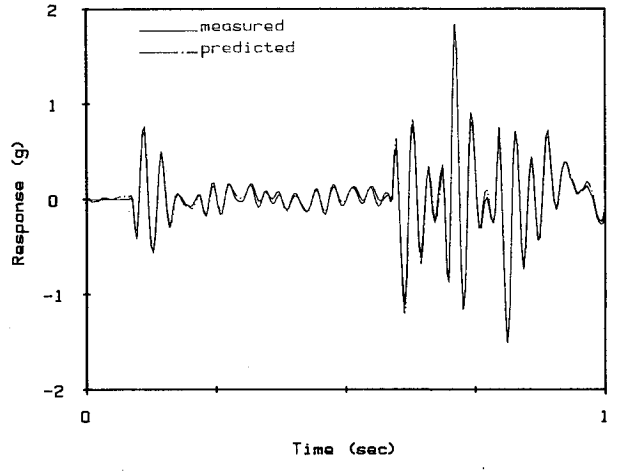

Fig. 10 Response and prediction of response (case B).

The reduced-order model is in a causal form and can be used to predict vibration. The model was used along with the input shown in Fig. 4 to predict the acceleration response of the aircraft wingtip. A portion of the predicted response is compared with the measured response in Fig. 10. The prediction is highly accurate for this arbitrary input. It is difficult to differentiate between the actual and the predicted response on the plot. The power in the prediction error is about 1% of the power in the measured signal.

Conclusions and Discussion

The reduced backward method for the parameter identification of discrete time-series models was presented. The method uses model overspecification to help reduce errors. Because of the way in which the overspecified model is identified, portions of the model can be separated as either system or computational components. The computational portions can be automatically identified and eliminated resulting in a reduced-order model. The reduced model provides highly accurate vibration prediction. The procedure is fully automatic starting with time-domain measurement data and providing a parametric system model.

As with any procedure, there are certain limitations to the method. First due to the approximate nature of the minimum norm solution, a computational pole may be classified as a system pole. In this case, the reduced model will be larger than necessary and contain an extra mode. The contribution of the mode should be minimal, since it should be smaller than the contributions of the system modes. The additional computational expense should also be limited. In any case, the reduced model would be preferred over an overspecified LS ARMA model that has not been reduced. The other possible limitation occurs when a system pole is classified as a computational pole. In such a case, the reduced model will be smaller than necessary and will be missing an actual mode. Such an omission occurs for a mode where the magnitude of the response contribution from that mode is on the order of magnitude of the noise. The contribution of this mode may not be very important, but additional study of such a case is needed. The cause of either failure is a result of noise bias perturbation. The noise perturbs the poles to cross the unit circle. There are two possible solutions that can be used to reduce this perturbation, increase the overspecification order to reduce the amount of perturbation, or acquire better data.

The results presented in this paper are encouraging because the techniques were applied using experimental data from a complex composite structure. The sailplane, fabricated from fiberglass, wood, and plastic, is an infinite degree-of-freedom, continuous structure with some of the nonlinearities and modal density properties that would be expected on a practical structure. The time-series data, experimentally acquired, presumably contains "colored" noise, nonlinearities, and higher-mode responses. Yet, the reduced backward method was able to automatically generate a low-order model from this data. The reduced-order model had similar modal characteristics to an averaged frequency-domain representation utilizing only one-twenty-fifth of the data. This reduced model was used to provide transient-response prediction with a high degree of accuracy.

Some issues directly related to the accuracy of the identification procedures were not addressed because of the limited scope of this paper. These concerns are related to the required characteristics of the input time series and the data-acquisition procedures used to acquire the input–output time histories. The input has to have a certain statistical character and some of the characteristics of the input are given by Refs. 4 and 11. The data-acquisition considerations include selection of sample rate, data filtering, measurement devices, and the data-acquisition system; for a discussion see Ref. 10.

Appendix: Minimum Norm Solution Properties

A number of properties of the minimum norm solution are demonstrated by example in Ref. 12. In this Appendix, these properties are demonstrated in an analytic derivation. The basis for the analytical development is outlined in Ref. 8. The derivations given in this Appendix are an expansion of those given by Ref. 8 using notes provided by a series of lectures from Ref. 13.

The norm of a vector $\boldsymbol{c}$ is defined as

$$||\boldsymbol{c}|| = \sum_{n=0}^{m} c_n^2 \qquad \text{(A1)}$$

where

$$c = [1, c_1, \ldots, c_m]^T \tag{A2}$$

If c is the solution to a singular problem, there are an infinite number of possible solutions. The minimum norm solution is the one solution that has the minimum norm of all possible solutions.

A polynomial can be written using the solution vector components as coefficients:

$$C(z) = 1 + c_1 z + c_2 z^2 + \ldots + c_m z^m \tag{A3}$$

A conjugate polynomial is of the form

$$C'(z) = 1 + c_1 z^{-1} + c_2 z^{-2} + \ldots + c_m z^{-m} \tag{A4}$$

The usefulness of the conjugate polynomial is evident when the two polynomials are multiplied together:

$$C(z)C'(z) = \sum_{n=0}^{m} c_n^2 + D(z) \tag{A5}$$

where $D(z)$ contains all of the terms that are a function of z. Now, if one evaluates a line integral of Eq. (A5) along the unit circle, the result is

$$\frac{1}{2\pi}\int_0^{2\pi} C(e^{j\theta})C'(e^{j\theta})\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} |C(e^{j\theta})|^2\mathrm{d}\theta = \sum_{n=0}^{m} c_n^2 \tag{A6}$$

Equation (A6) shows that the norm of a vector can be found by finding the "norm" of a corresponding polynomial. It would seem that the simple definition of a vector norm has been replaced by a more complicated line integral. But the line integral of a complex function has some fundamental properties that are very useful.

The $C(z)$ polynomial can be factored into two polynomials:

$$C(z) = C_1(z)C_2(z) \tag{A7}$$

Since c is a singular solution, one factor of the corresponding polynomial is fixed and the other factor is variable. Let $C_1(z)$ represent the fixed polynomial. One can factor $C(z)$ into the function

$$C(z) = f_i(z)\{z - z_i\} \tag{A8}$$

where z_i represents the ith root of $C_2(z)$. Substitution of Eq. (A8) into Eq. (A6) results in the expanded form

$$\int_0^{2\pi} |C(e^{j\theta})|^2\mathrm{d}\theta = \int_0^{2\pi} |f_i(e^{j\theta})|^2\mathrm{d}\theta - \int_0^{2\pi} |f_i(e^{j\theta})|^2 e^{j\theta} z_i{}^* \mathrm{d}\theta$$
$$- \int_0^{2\pi} |f_i(e^{j\theta})|^2 e^{-j\theta} z_i \mathrm{d}\theta + \int_0^{2\pi} |f_i(e^{j\theta})|^2 z_i{}^* z_i \, \mathrm{d}\theta \tag{A9}$$

The minimum of Eq. (A9) can be found by differentiation with respect to z_i and equating the result to zero. The result gives the expression

$$z_i = \frac{\int_0^{2\pi} |f_i(e^{j\theta})|^2 e^{j\theta} \, \mathrm{d}\theta}{\int_0^{2\pi} |f_i(e^{j\theta})|^2 \, \mathrm{d}\theta} \tag{A10}$$

The magnitude of the numerator is less than the magnitude of the denominator in Eq. (A10). This fact can be shown through the general expression given below:

$$\left|\int_c p(z)e^{jz}\,\mathrm{d}z\right| \le \int_c |p(z)|\,|e^{jz}|\,\mathrm{d}z = \int_c |p(z)|\,\mathrm{d}z \tag{A11}$$

The result is

$$|z_i| \le 1 \tag{A12}$$

The magnitude of all of the roots of $C_2(z)$ are less than unity for a minimum norm solution.

Now consider the case where the solution vector is partitioned into two vectors:

$$c = [1 a_1 a_2 \ldots a_m b_0 b_1 \ldots b_m]^T \tag{A13}$$

The norm of the vector is the sum of the norms of the two smaller vectors:

$$||c|| = \sum_{n=0}^{2m} c_n^2 = \sum_{n=0}^{m} a_n^2 + \sum_{n=0}^{m} b_n^2 \tag{A14}$$

The norm of the solution vector can be replaced by two line integrals:

$$||c|| = \frac{1}{2\pi}\int_0^{2\pi} |A(e^{j\theta})|^2\,\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} |B(e^{j\theta})|^2\,\mathrm{d}\theta \tag{A15}$$

The corresponding polynomials can be factored as

$$A(z) = A_1(z)A_2(z) \tag{A16}$$

$$B(z) = B_1(z)B_2(z) \tag{A17}$$

If the ratio of the polynomials is set and is equal to the ratio between $A_1(z)$ and $B_1(z)$, as in the case of the identification of an overspecified model for noise-free data, then $A_2(z)$ is equivalent to $B_2(z)$. The two polynomials can be factored as

$$A(z) = g_i(z)\{z - z_i\} \tag{A18}$$

$$B(z) = h_i(z)\{z - z_i\} \tag{A19}$$

where z_i represents the ith root of $A_2(z)$. When Eq. (A15) is expanded and minimized with respect to z_i, the result is

$$z_i = \frac{\int_0^{2\pi} |g_i(e^{j\theta})|^2 e^{j\theta}\,\mathrm{d}\theta + \int_0^{2\pi} |h_i(e^{j\theta})|^2 e^{j\theta}\,\mathrm{d}\theta}{\int_0^{2\pi} |g_i(e^{j\theta})|^2\,\mathrm{d}\theta + \int_0^{2\pi} |h_i(e^{j\theta})|^2\,\mathrm{d}\theta} \tag{A20}$$

The magnitude of the roots of $A_2(z)$ can be shown to be less than unity using an argument based upon Eq. (A11).

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